DEPARTMENT OF APPLIED MATHEMATICS

Indian Institute of Science

BANGALORE – 560 012, INDIA

DR.A.V.GOPALAKRISHNA
CONVENER, MATHEMATICS OLYMPIAD

Ref. No. AM/MO/89-183. *Date.* 17th August 1989

Dear Student,

INMO – 89 will be held in Lecture Hall L – 1, Department of Applied Mathematics, Indian Institute of Science, Bangalore on Sunday, September 24, 1989 between 1 pm and 4 pm. Based on your performance in the Olympiad conducted by IISc, you are eligible to participate in INMO – 89. Please let me know whether you will take part in INMO – 89.

Outstation candidates will be paid two-way sleeper second class train fare or bus fare (KSRTC) from their place of study to Bangalore plus Rs.50/- as incidental expenses.

Looking forward to hearing from you.

Yours sincerely,

AVGopalakrishna
17.8.'89

(A.V.GOPALAKRISHNA)

To

Mr.L.Shyamal
D-206, IISc Campus
Indian Institute of Science
Bangalore 560012

~~BOOKS THAT SHOULD BE DISTRIBUTED TO SCHOOLS ON A VERY EXTENSIVE BASIS BY NBHM~~

List of books suitable for the preparation of INMO

(1) The entire 'LITTIE MATHEMATICS LIBRARY' (MIR Publishers)

(2) Books from MIR Publisers (Modenov; Shklarsky; Krechmar; Sharygin)

(3) The Hungarian Problem Books, I & II (based on EOTVOS)
(NML - New Mathematical Library series published by the Mathematical Association of America).*

(4) The Olympiad Collections (1958-1979, 1978-1985 - IMO), NML series

(5) Barnard & Child : Higher Algebra
Loney : Trigonometry
Durell : Modern Geometry
G. Andrews : Number Theory
Niven & Zuckerman : Number Theory
F. Harary : Graph Theory
Burnside & Panton : Theory of Equations
Indian publications

(6) The Contest Problem Books, I, II, III NML series

(7) Mathematics for Pre-college Students, Baranov, Bogatyrev and Bokovnev
MIL Publishers.

(8) Ross Honsberger : Mathematical Gems, I & II, Mathematical Assn. of America

(9) The William Lowell Putnam Mathematical Competition - Problems and Solutions 1938-1964, 1965-85 (2 vols). Mathematical Assn. of America.

(10) Maxima & Minima Without Calculus by Ivan Niven - Mathematical Assn. of America

(11) Problems in Elementary Mathematics for Home Study, Arithmetics, Algegra, Geometry and Trigonometry - MIR Publishers. (Authors : N. Anatonov, M.Vygodsky, V.Nikitin, A. Sankin.

(12) Ross Honsberger (Ed.) : Mathematical Plums, Math. Assn. of America.

(13) Mathematical Morsels - Ross Honsberger (Ed.), Math. Assn. of America

(14) The Contest Problem Book, Vols. I, II, III & IV (by Charles T. Salkind et al.)

(15) D.O. Shikyankv, N.M. Chentsov & I.M. Vaglom : Selected Problems and Theorems in Elementary Mathematics - MIR Publishers.

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23/6/89

* Mathematical Association of America, 1529, 18th St., N.W., Washington DC 20036, USA.

NBHM (DAE)

INMO—1989

Time : Three Hours] [Maximum Marks : 100

Attempt as many questions as you possibly can.

1. Prove that the polynomial

$$f(x)=x^4+26x^3+52x^2+78x+1989$$

cannot be expressed as a product

$$f(x)=p(x)\,q(x)$$

where $p(x)$, $q(x)$ are both polynomials with integral coefficients and with degree < 4.

2. Let a, b, c, d be any four real numbers, not all equal to zero. Prove that the roots of the polynomial

$$f(x)=x^6+ax^3+bx^2+cx+d$$

cannot all be real.

3. Let A denote a subset of the set $\{1, 11, 21, 31, \ldots 541, 551\}$, having the property that no two elements of A add up to 552. Prove that A cannot have more than 28 elements.

4. Determine, with proof, all the positive integers n for which :

   (i) n is not the square of any integer ; and

   (ii) $[\sqrt{n}]^3$ divides n^2.

(Notation : $[x]$ denotes the largest integer that is less than or equal to x).

5. Let a, b, c denote the sides of a triangle. Show that the quantity,

$$\frac{a}{(b+c)}+\frac{b}{(c+a)}+\frac{c}{(a+b)}$$

must lie between the limits 3/2 and 2. Can equality hold at either limit ?

6. Triangle ABC is scalene with angle A having a measure greater than 90 degrees. Determine the set of points D that lie on the extended line BC, for which

$$|AD| = \sqrt{(|BD|\ |CD|)}$$

where $|BD|$ refers to the (positive) distance between B and D.

7. Let ABC be an arbitrary acute angled triangle. For any point P lying within this triangle, let D, E, F denote the feet of the perpendiculars from P onto the sides AB, BC, CA respectively. Determine the set of all possible positions of the point P for which the triangle DEF is isosceles. For which position of P will the triangle DEF become equilateral ?

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Shiyamal

Total Pages : 2

NBHM (DAE)

INMO—1988

Time : Three Hours] [Maximum Marks : 100

Attempt all questions.

1. Let $m_1, m_2, m_3, \ldots\ldots, m_n$ be a rearrangement of the numbers $1, 2, \ldots\ldots, n$. Suppose that n is odd. Prove that the product $(m_1-1)(m_2-2), \ldots\ldots, (m_n-n)$ is an even integer.

2. Prove that the product of 4 consecutive natural numbers cannot be a perfect cube.

3. Five men, A, B, C, D, E are wearing caps of black or white colour without each knowing the colour of his cap. It is known that a man wearing black cap always speaks the truth while the ones wearing white always tell lies. Of they make the following statements, find the colour worn by each of them : W B

   A : I see three black caps and one white W

   B : I see four white caps W

   C : I see one black cap and three white B

   D : I see four black caps. W

   B → E

4. If a and b are positive and $a+b=1$, prove that

$$\left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2 \geqslant 12\frac{1}{2}.$$

Shyamal

5. Show that there do not exist any distinct natural numbers a, b, c, d such that

$$\boxed{\begin{array}{c} a^3+b^3=c^3+d^3 \\ a+b=c+d. \end{array}}$$

6. If $a_0, a_1, \ldots\ldots, a_{50}$ are coefficients of the polynomial $(1+x+x^2)^{25}$ show that $a_0+a_2+a_4+\ldots\ldots+a_{50}$ is even.

7. Given an angle Q B P and a point L outside the angle Q B P. Draw a straight line through L meeting BQ in A and BP in C such that the triangle A B C has a given perimeter.

8. A river flows between two houses A and B, the houses standing some distances away from the banks. Where should a bridge be built on the river so that a person going from A to B, using the bridge to cross the river may do so by the shortest path ? Assume that the banks of the river are straight and parallel, and the bridge must be perpendicular to the banks.

9. Show that for a triangle with radii of circum-circle and in-circle equal to R, r respectively, the inequality $R \geqslant 2r$ holds.

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jamal

NBHM (DAE)

INMO—1987

Time : Three Hours] [Maximum Marks : 100

Attempt all questions.

1. Given m and n as relatively prime positive integers greater than one, show that $\log_{10}m/\log_{10}n$ is not a rational number.

2. Determine the largest number in the infinite sequence

   $1, \sqrt[2]{2}, \sqrt[3]{3}, \sqrt[4]{4}, \ldots\ldots, \sqrt[n]{n}, \ldots\ldots\ldots$ $\sqrt[e]{e}$

3. Let T be the set of all triplets (a, b, c) of integers such that $1 \leqslant a < b < c \leqslant 6$. For each triplet (a, b, c) in T, take the number $a \times b \times c$. Add all these numbers corresponding to all the triplets in T. Prove that the answer is divisible by 7.

4. If x, y, z and n are natural numbers, and $n \geqslant Z$ then prove that the relation $x^n + y^n = z^n$ does not hold.

5. Find a finite sequence of 16 numbers such that :

   (a) it reads same from left to right as from right to left.

   (b) the sum of any 7 consecutive terms is -1.

   (c) the sum of any 11 consecutive terms is $+1$.

6. Prove that if coefficients of the quadratic equation $ax^2 + bx + c = 0$ are odd integers, then the roots of the equation cannot be rational numbers.

7. Construct the $\triangle ABC$, given h_a, h_b (the altitudes from A and B) and m_a, the median from the vertex A.

8. Three congruent circles have a common point O and lie inside a given triangle. Each circle touches a pair of sides of the triangle. Prove that the incentre and the circumcentre of the triangle and the common point O are collinear.

9. Prove that any triangle having two equal internal angle-bisectors (each measured from a vertex to the opposite side) is isosceles.

Shyamal

NBHM (DAE)

INMO—1986

Time : Three Hours] [Max. Marks : 100

Attempt all questions.

1. A person who left home between 4 p.m. and 5 p.m. returned between 5 p.m. and 6 p.m. and found that the hands of his watch had exactly exchanged places. When did he go out ?

2. Solve :

$$\log_2 x+\log_4 y+\log_4 z=2$$

$$\log_3 y+\log_9 z+\log_9 x=2$$

$$\log_4 z+\log_{16} x+\log_{16} y=2$$

3. Two circles with radii a and b respectively touch each other externally. Let c be the radius of a circle that touches these two circles as well as a common tangent to the two circles Prove that

$$\frac{1}{\sqrt{c}}=\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}$$

4. Find the least natural number whose last digit is 7 such that it becomes 5 times larger when this last digit is carried to the beginning of the number.

$$\begin{array}{r} 142857 \\ \times\ 5 \\ \hline 714285 \end{array}$$

5. If $P(x)$ is a polynomial with integer coefficients and a, b, c three distinct integers, then show that it is impossible to have $P(a)=b$, $P(b)=c$, $P(c)=a$.

6. Construct a quadrilateral which is not a parallelogram, in which a pair of apposite angles and a pair of opposite sides are equal.

7. If a, b, x, y are integers greater than 1, such that a and b have no common factor except 1 and $x^a = y^b$, show that $x = n^b$, $y = n^a$ for some integer n greater than 1.

8. Suppose $A_1, \ldots, A_6$ are six sets each with four elements and $B_1, \ldots, B_n$ are n sets each with two elements. Let $S = A_1 \cup A_2 \cup \ldots \cup A_6 = B_1 \cup B_2 \cup \ldots \cup B_n$. Given that each element of S belongs to exactly four of the A_i's and to exactly three of the B_j's, find n.

9. Show that among all quadrilaterals of a given perimeter the square has the largest area.

Mathematics Olympiad: Problem - solving sessions
(S.Shirali)

These remarks are meant to be prefatory.

Problem - solving is the life - blood of mathematics. Vast areas of mathematics have resulted from the attempts to solve difficult problems. Reflection on just a few instances will reveal the truth of this statement:- Newton's invention of the calculus and the problem of gravitational attraction: Galois' field theory and the unsolvability of the general quintic equation; the Kummer/Dedekind theory of 'ideal' numbers and 'Fermat's last theorem'; Hilbert's twenty three problems (publicly posed in 1900). At the same time there must be theory building.

The problems posed here are for the most part fairly challenging. Several of the problems have been selected from past Olympiad papers. The collection does not represent a random choice - rather, they have been chosen to illustrate certain recurring themes in mathematics - theorem-proving strategies, so to speak. There is some leaning towards problems of a more combinatorial nature - reflecting a purely personal inclination of this author.

PART 1 :- 'Appetizers'

1. Show that there is essentially only one way of filling in the cells of a 3 x 3 grid with the numbers 1,2......7,8,9 so as to form a magic square.

2. Let ABC be an arbitrary (scalene) triangle, with M the mid-point of BC, so that the median AM cuts the larger triangle into two triangles of equal area. Find the simplest possible way of cutting up triangle ABM and re-assembling the pieces so as to fit exactly over triangle AMC.

3. Imagine three unequal circles "pushed" together in such a way that each circle is tangent (externally) to the other two circles. Let the common tangent to each pair of circles be drawn. Show that the three tangents meet in one point (i.e., are concurrent).

...2

4. Call a natural number "sorted" if its digits are strictly in descending or in ascending order (e.g. 124 or 1689 or 752 but not 1223 or 324). How many sorted numbers are there?

5. Find the smallest perfect square whose digits start with 1111111........ 333333

6. Consider the sequence of prime numbers 2,3,5,7, ... and form the "cumulative sum" sequence (also called the "partial sums") :- 2,5,10,17,28,41,58, ... Prove that between any two consecutive terms of this sequence there is always at least one perfect square.

PART II :- "The main course"

(A) NUMBER THEORY:

A1:- Investigate the equation $Y^2 = 1 + x + x^2$ for integral solutions (x,y).

A2:- Find all solution in integers to $Y^2 = 1 + x + x^2 + x^3$

A3:- Prove that the only integral pairs (x,y) satisfying $Y^2 = 1 + x + x^2 + x^3 + x^4$ are

$(-1, 1)$, $(0,1)$ and $(3,11)$

A4:- Find the most general solution in positive integers x,y,z to

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{z}$$

A5:- "Formats' last theorem" (unproved as of today) states that the equation $x^n + y^n = z^n$ has no solution in psoitive integers if n is a natural number greater than 2.

Prove the following (milder) version of the theorem:- the system

$$x^n + y^n = z^n$$

$$2 < z < n$$

has no solution in positive integers.

A6:- Establish the following equality (one of Ramanujans' more "elementary" results) :-

$$\frac{q}{1-q} + \frac{q^3}{1-q^3} + \frac{q^5}{1-q^5} + \frac{q^7}{1-q^7} + \ldots\ldots\ldots$$

$$= \frac{q}{1-q} + \frac{q^3}{1-q^2} + \frac{q^6}{1-q^3} + \frac{q^{10}}{1-q^4} + \ldots$$

Where the exponents of the numerators on the RHS are the "triangular numbers" (1,3,6,10,15,21,28,.......)

A7:- Consider a long but finite row of houses, sequentially numbered 1,2,3,..... from one end to the other. I stay in a house on this row which a has the special property that the sum of all the houses numbers to my <u>left</u> exactly equals the sum of all the house numbers to my <u>right</u>. What could my house number be? Uncover **<u>all</u>** possible solutions.

A8:- The numbers 1, 25, 49 are all perfect squares and also they form an arithmetic progression (i.e. 25 = (1 + 49)/2). Show that there are infinitely many such number triples and find a scheme that can generate <u>all</u> such triples. (Note:- Since (1, 25, 49) is such a triple, so is (4, 100, 196); but this contains a common factor of 4 and is thus not so 'interesting'. We need a way of generating <u>all</u> possible such triples that have no common factor.)

What if we ask for <u>four</u> squares that form an arithmetic progression? Exhibit an instance of this or else prove that no such instance exists.

A9:- (i) Prove that the only positive integral solution to

$$2^x - 3^y = 1$$

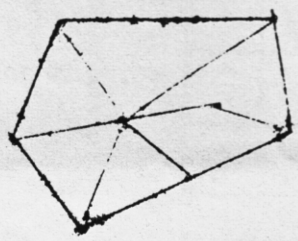
is $(x,y) = (2,1)$

(ii) Prove that the only positive integral solution to

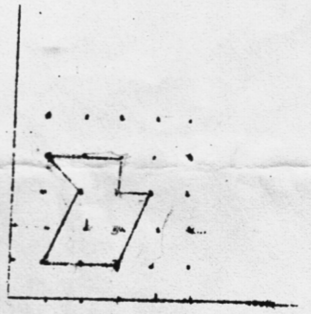
$$3^x - 2^y = 1$$

is $(x,y) = (2,3)$

(B): COUNTING (AND RELATED MATTER)



B1:- A configuration of the type shown at right is termed a 'network'; it has edges whose end-points are vertices (e.g., $\bullet\!\!-\!\!-\!\!\bullet$), and the edges can enclose a face. The network show has $E = 14$ edges, $V = 9$ vertices and $F = 6$ faces. Note that $V - E + F = 1$ Prove that $V - E + F = 1$ for any network (a result due to L.Euler)



B2:- Consider a rectangular cartesian plane and a simple (see appendix for definitions) polygonal figure on it whose vertices are all lattice points (see appendix for definitions). Let the figure enclose I lattice points in its interior and have B lattice points on its boundary; let its area be A. For instance, the figure shown has $I = 2$, $B = 9$, $A = 5\frac{1}{2}$ Prove that the equality

$$A = I + \frac{B}{2} - 1$$

always holds (a marvellous result due to Pick).

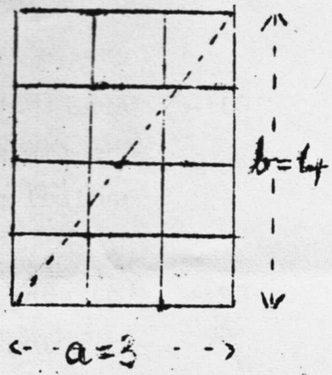
B3:- Let n points be located on the periphery of a circle and let all possible chords joining these points be drawn. Assume that the points have been placed somewhat irregularly, so that no three chords pass through any one point (apart, of course, from the original points chosen). Determine:-

(a) the total number of points of intersection;

(b) the number of distinct regions into which the circle is cut up (e.g. 8 regions for $n = 4$).

(The assumption of non-concurrency of the chords means of course, that we are determining the largest possible number in (a), (b), above.)

B4:- Consider a rectangular grid composed of cells, 'a' on one side and 'b' on the other (a, b being positive integers). When a main diagonal is drawn, it passes through the interiors of a number of cells, say N (e.g. N = 6 in the figure). Compute N <u>explicitly</u> as a function of a, b.



N(a,b)

B5:- Let a, b be positive integers having no factors in common (apart from 1). Denote by A_n the number of non-negative integral solutions (x,y) to :-

$$ax + by = n$$

Prove:-

(i) $A_{ab-a-b} = 0$

(ii) $A_{n+ab} = A_n + 1$

(iii) $A_n = \left[\frac{n}{ab}\right]$ or $\left[\frac{n}{ab}\right] + 1$

(for the meaning of the symbol, [ ] , see the appendix).

(iv) $A_n > 0$ if $n > ab - a - b$

B6:- In how many different ways can you give change (in coins) for one rupee? Assume that the coins are available in these denominations;- 1 p., 5 p., 10 p., 25 p., 50 p.

.....6/-

(C) ESTIMATION THEORY

C1:- Show that the equation $x^{1000} = 10^{x}$ has two solutions. Estimate the positive solution to two decimal places.

C2:- (a) Show why the output of the following algorithm gives the logarithm of any given number x to base 10, directly in binary notation. The algorithm is presented in computer flowchart notation:-

Input: a given number x, $1 < x < 10$

Algorithm: (in 3 steps)

Step 1:- Let $x := x^2$
Go to step 2

Step 2:- If $x < 10$, print '0' and return to step 1.
If $x \geqslant 10$, print 1 and go to step 3.

Step 3:- Let $x := x/10$
Return to step 1.

e.g. the input number x = 5 gives the output sequence (1011001.....) so that, in binary notation

$$\log_{10} 5 = 0.1011001 \ldots.$$

(Note that this method, though of interest, is definitely not a practical tool for evaluating logarithms!)

(b) Show how the above algorithm can be modified to give the logarithm of x to base 10 in ordinary decimal notation.

C3:- Given any positive number N and an estimate, x, ($x > 0$), for the positive square root of N (but $x \neq \sqrt{N}$), show that a strictly better estimate is given by

$$x' = x - \frac{x^2 - N}{2x}$$

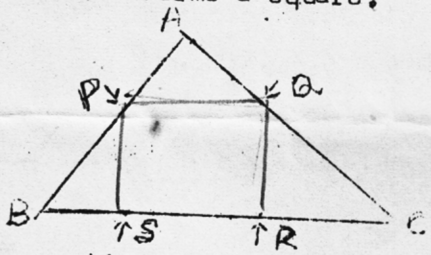
(a result due to Newton). Does x' always err in the same direction as x ?

(D): GEOMETRY

D1:- Three circles intersect each other mutually, i.e. each passes through the interiors of the other two circles. For each pair of circles, the common chord is drawn - i.e. the straight line passing through their points of intersection. Prove that the three straight line pass through one point.

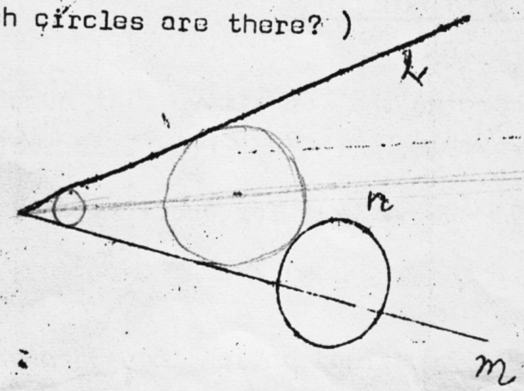
D2:- On each side of an arbitrary triangle ABC is constructed an equilateral triangle, facing outwards. Let the centres of these three equilateral triangles be x, y, z. Prove that x,y,z are themselves the vertices of an equilateral triangle.
(This result is known as Napoleon's theorem - but it is doubtful whether Napoleon really hit upon it !)

D3:- Given:- an arbitrary acute-angled triangle ABC. Find a general procedure for locating points P,Q,R,S (P on AB, Q on AC, R and S on BC) So that PQRS forms a square.



magnify square at corner

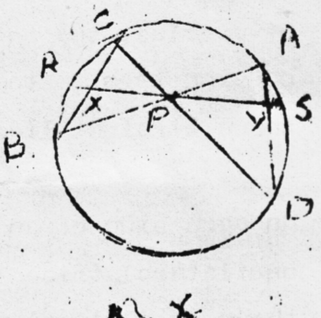
D4:- Given:- two intersecting lines l,m meeting at an arbitrary angle and an arbitrarily placed circle n. How would you construct a circle that touches (i.e. is tangent) to each of l,m,n? (How many such circles are there?)



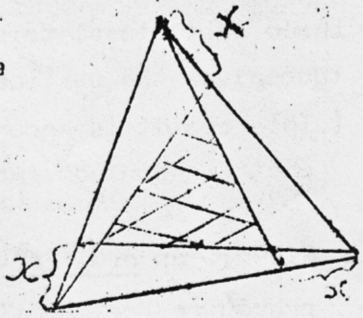
D5:- Given:- an arbitrary acute angled triangle ABC. Determine points P, Q, R on the sides AB, BC, CA of the triangle so that triangle PQR has the shortest possible perimeter. What happens when ABC is not acute angled?

D6:- Let AA' denote a fixed chord of a circle, XOX' a variable diameter and Z be the intersection AX.A'X'. Determine the locus of all possible Z as XOX' varies.

D7:- ("The Butterfy Theorem") In the figure P is the midpoint of a chord RS of the circle. AB, CD are chords through P with BC, AD intersecting RS in X,Y. Prove that P is the midpoint of XY

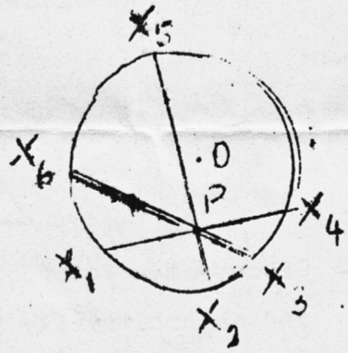


D8:- Shown at right is an equilateral triangle cut up symme-trically. The side of the large equilateral triangle is 1. Determine the area of the central triangle, f(x), as a function of x.



D9:- Shown at right is a circle, enclosing point P and three chords X_1X_4, X_2X_5, X_3X_6 that pass through P. If each of the six angles X_1PX_2, X_2PX_3, ..., X_6PX_1 equals 60°, prove that

$$PX_1 + PX_3 + PX_5 = PX_2 + PX_4 + PX_6$$



(E): PLAYING WITH FUNCTIONS

E1: Let f be a function mapping the { positive real numbers } into the { positive reals } , with the properties that

(i) $f(x.f(y)) = y.f(x)$ for every $x, y \in R$

(ii) $f(x) \rightarrow 0$ as $x \rightarrow \infty$

Identify with proof all functions having these properties.

E2:- Let f be a function mapping N into N, where N is { natural numbers }, with the property that

$$f(n+1) > f(f(n)) \quad \text{for every } n \text{ in } N$$

Show that there is just one function f that satisfies this restriction and identify the function.

E3:- Let f denote the function that maps x (a real number) onto the integer **closest** to x, e.g. $f(2.3) = 2$, $f(3.81) = 4$, $f(6.5) = 7$. We use the convention that numbers with fractional part equal to $\frac{1}{2}$ (= 0.5) are rounded up ($f(3.5) = 4$, etc)

Now you must be familar with the famous equality (summation of an infinite geometric series),

$$1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots\ldots \text{ (to infinity).}$$

So for any integer N,

$$N = \frac{N}{2} + \frac{N}{4} + \frac{N}{8} + \frac{N}{16} + \ldots\ldots$$

Prove that this equlity remains valid even with the terms on the r.h.s. replaced by their f-values, i.e.

$$N = f\left(\frac{N}{2}\right) + f\left(\frac{N}{4}\right) + f\left(\frac{N}{8}\right) + f\left(\frac{N}{16}\right) + \ldots\ldots$$

(e.g. $10 = 5 + 3 + 1 + 1 + 0 + 0 + \ldots\ldots$)

E4:- Let f : Real numbers → Real numbers such that

$$\text{(i)} \quad f(x+y) = f(x) + f(y)$$

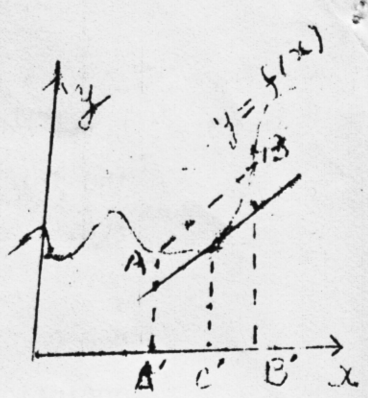
$$\text{(ii)} \quad f(x.y) = f(x) \,.\, f(y),$$

for every possible choice of x,y

Identify all functions that have this property.

E5:- Show that therecannot exist any function f mapping {natural numbers} into {natural numbers} (i.e. N → N) for which

$$f(f(n)) = n + 2001 \quad \text{for every } n.$$



E6:- Consider the graph of any function $f : R \rightarrow R$. Assume that f is continuous and smooth (i.e. possesses no 'sharp corners'). If A,B are two arbitrary and distinct points on the curve, it is intuitively and visually clear (and this result is also rigourously provable) that there must be a point C on the curve in-between A and B at which the tangent to the curve is parallel to the chord AB. Naturally the exact location of C depends on the locations of A and B.

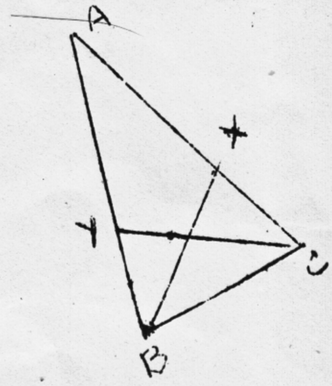
Find all possible curves for which (considering the projection A',C',B' on the x-axis), C' is always the midpoint of A'B'.

(F): A POT - POURRI

F1:- Let x be any irrational number (e.g. $x = \sqrt{2}$); let e be any given positive number, however small (e.g. e = 0.001). Prove that there is some multiple of x that lies within a distance e of an integer, i.e. there exist integers m,n with

$$|mx - n| < e$$

F2:- Consider any set consisting of ten distinct integers, each drawn from the range 1...100. Prove that the set must contain two disjoint subsets the total sum of whose elements is the same.



F3:- Triangle ABC is shown at right with its internal anglebisectors drawn. If BX = CY, show that the triangle ABC must be isosceles. (This is a famous and old problem, generally known as the Steiner - Lehmus theorem - and its rather trickier than it appears.)

F4:- Ramanujan once posed this problem:- Evaluate the infinite continued-root

$$1\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+5\sqrt{\cdots}}}}} \quad \text{.....etc}$$

How would you tackle it?

F5:- Prove that for any integer $n \geqslant 1$, there exists a power of 2 whose last n digits (when written in the ordinary denary form) are all 1's and 2's.

Appendix

Simple polygon: a polygon for which any two points in its interior can be joined by a curve which does not intersect the boundary.

Lattice points: those points in the plane which have coordinates (i,j) where i.j take any integral value zero positive or negative.

$\left[\, k \,\right]$ = the greatest integer not greater than k.

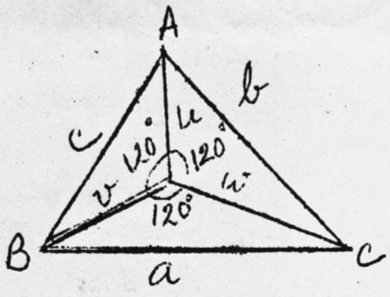
yamal

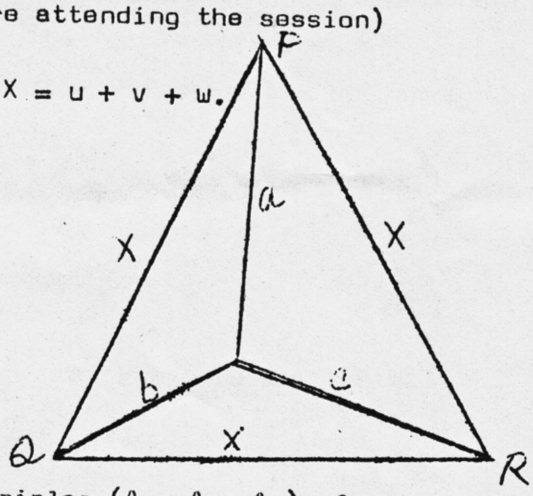
PROBLEM SOLVING SESSION

by Dr.C.R.Praneshachar

(Please attempt these problems before attending the session)

1. In the two triangles given below, show that $X = u + v + w$.





2. Determine with proof the number of ordered triples (A_1, A_2, A_3) of sets which have the property that (i) $A_1 \cup A_2 \cup A_3 = \{1,2,3, \ldots 10\}$ (ii) $A_1 \cap A_2 \cap A_3 = \emptyset$, where $\emptyset$ denotes the empty set.

3. Which is greater:

$$1987^{1987} \quad \text{or} \quad (1987!)^2$$

4. There are 17 points in a plane, no three of which are collinear. All pairs of points are joined by lines coloured red, blue or green. Show that there is a monochromatic triangle, i.e., a triangle all the three sides of which have the same colour.

5. Sol for x,y,z ($xyz \neq 0$)

$$\left.\begin{aligned} x + y + z &= -xyz \\ x^2 + y^2 + z^2 &= \frac{3}{2} x^2 y^2 z^2 \\ x^3 + y^3 + z^3 &= -x^3 y^3 z^3 \end{aligned}\right]$$

6. Evaluate

$$1^4 \binom{n}{1} + 2^4 \binom{n}{2} + 3^4 \binom{n}{3} + \ldots\ldots + n^4 \binom{n}{n}$$

Where $\binom{n}{r}$ denotes the number $\dfrac{n!}{r!\,(n-r)!}$

7. If $p_1, p_2, p_3, p_4, p_5, p_6$ are the probabilities of getting 1,2,3,4,5,6 in the throw of an uneven die and $q_1, q_2, q_3, q_4, q_5, q_6$ are the corresponding probabilities for a second die, is it possible to force the values of p's and q's such that in any throw of the two dice the sum of the outcomes has the same probability?

8. Show that the only integral values of m for which $1 + m + m^2 + m^3 + m^4$ is a perfect square are $m = 0$, $m = -1$ and $m = 3$.

9. Let r be a positive rational less than 1. Show that r can be written as the sum of distinct elements of the set $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\ldots\ldots\right\}$.

10. There are 11 positive integers $a_1, a_2, \ldots\ldots, a_{11}$ such that for each a_i ($1 \leq i \leq 11$), the remaining 10 numbers can be split into two sets of 5 numbers each having equal sums. Show that all the 11 numbers are in fact equal.

11. If the polynomial

$$p(x) = (x-m_1)(x-m_2)(x-m_3)(x-m_4)(x-m_5),$$ where m_1, m_2, m_3, m_4, m_5 are 5 distinct integers, has exactly K non zero coefficients, find with proof a set of integers $m_1, m_2 \ldots\ldots m_5$ for which this number K is minimum.

12. Show that any number N can be written in the 'modified base 3', as

$$N = \sum_{j=0}^{K-1} a_j 3^j ,$$ where a_j is -1, 0 or 1.

Is the representation unique?

13. There is a finite set S of coplanar points having the property that on the line joining any two points of the set S, there lies a third point of the set S. Prove that all the points of the set S are collinear.

14. There is strictly increasing sequence $\{f(n)\}_{n=1}^{\infty}$ of positive integers with the following property: $f(2)=2$ and $f(mn)= f(m)f(n)$ whenever m and n are relatively prime. Show that $f(n) = n$ for each n.

15. Find all polynomials P (x) satisfying $P(0) = 0$ and $P(x^2+1) = [P(x)]^2 + 1$.

$1 \in N$
$n \in N$
$n+1 \in N$
$\{n\} = N$

16. Solve the recurrence relation:

$$u_n = (n-1)\,(u_{n-1} + u_{n-2}), \quad n \geqslant 3;$$

$$u_1 = 0, \quad u_2 = 1.$$

Hence evaluate $\displaystyle\lim_{n \to \infty} \frac{u_n}{n!}$

17. Find a polynomial $A(x)$ of least degree such that $A(x)$ is divisible by $x^2 + 1$ and $A(x) - 1$ is divisible by $x^3 + 1$.

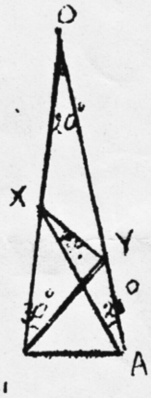
18. Three circles of equal radii r pass through a point O and have second points of intersection A, B, C. Prove that the circumcircle of $\triangle$ ABC is also of the same radius r.

19. In $\triangle\, OA_1A_2$, $\angle O = 20^\circ$

$OA_1 = OA_2$, $\angle OA_2X = 20^\circ$

$\angle OA_1Y = 30^\circ$. Show that

$\angle A_2XY = 30^\circ$.



~~20.~~ In an 8 X 8 chessboard ABCD, in how many ways can you go from A to C by moving always along horizontal and vertical lines but not going above the diagonal AC. You may touch AC. (Thus X is forbidden while Y & Z are not).

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